Product Boundary, Vertical Competition, and the Double Mark-up Problem

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Abstract

In the computing industry, computing systems typically consist of multiple components supplied by independent firms. An important feature of this industry is "co-opetition." In other words, firms must cooperate with each other in “making a system work,” but at the same time compete for dividing the industry profits. A firm may, by incorporating the functions of other firms’ components into its own product, make its product less dependent on other firms providing complementary components. When one of two firms expands its product boundary by incorporating the other firm’s function into its own product, we find that the firm’s expansion of the product boundary may increase not only its own profits, but also the other firm’s profits and social welfare.

Keywords: Vertical Competition, Product Boundary, Double Mark-up, Co-opetition

JEL Code: D4, D8, M3, L13
1. Introduction

In the computing industry, there has been no single dominant vertically integrated firm since 1990. Instead, the industry is characterized by vertical disintegration i.e., computer systems or platforms consist of many vertically related layers of components supplied by different firms. An important feature of this industry is ‘co-opetition,’ or ‘vertical competition’: firms cooperate with each other in “making a system work.” However, at the same time, they compete with each other for the industry profits.

To ensure a larger share of industry profits, firms try to make their products less dependent on providers of complementary products. As Bresnahan (1999) points out, a leading firm in each layer of a computing platform has the capability to challenge leading firms in other layers. Typically, a dominant firm in one layer attempts to include in its own products some functions provided in other layers. For example, in its early days, MS Windows did not include products such as WordPad, Internet Explorer (I.E.), Windows Media and Diskeeper (hard-disk defragmentation program), but, over time, has included these and other programs that were previously supplied by independent firms. Another example is secondary cache. Once a separate piece of hardware, secondary cache is now integrated into the Intel CPU. There is no exogenously given boundary between vertical layers, and firms constantly try to expand their product boundaries. As a result, the definition of layers and the delineation of product boundaries change continuously as a consequence of both technological innovation and vertical competition.

Analysing vertical competition is a key to understanding the computing industry. In this paper, we analyse vertical competition by explicitly modeling the concept of product boundary. We also analyse the effects of changing product boundaries on firms’ profits and social welfare. In our model, there are two firms, A and B, that provide complementary products A and B, respectively. In the beginning, each product does not provide any desired performance without the other product, and a consumer derives a performance quality of $q$ by combining both products. However, firm A can expand its product boundary, so that product A by itself provides a quality of $z$, which is lower than $q$. Regardless of the value of $z$, product B does not provide any function without product A, but enhances the latter's performance from $z$ to $q$. For instance, after Microsoft O/S
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included a basic text-editing program such as NotePad, MS O/S provided a basic text-editing function. However, if consumers want, they can buy an additional advanced word-processing program. Since consumers value product quality differently, some consumers buy only product A, while those with a higher valuation of quality buy both products.

The parameter z measures the boundary between the complementary products. For instance, they are strictly (and symmetrically) complementary if $z = 0$, but product B becomes redundant if $z = q$.

We first analyze the pricing game between firms A and B for a given $z$, $0 < z < q$. Then, we examine the effects of changes in $z$ on each firm’s pricing decisions and the firms’ profits. We find that firm A’s optimal price is not continuous in $z$; firm A’s profit increases continuously in $z$, while firm B’s profit exhibits an upward jump at a certain level of $z$, but afterwards decreases as $z$ increases.

Furthermore, we study how $z$ affects the product relationship between products A and B. If $z$ is low, then the firms’ individual profit-maximizing prices yield the standard characteristics of complementary goods. That is, each firm’s demand depends on the sum of the two prices. An interesting result in our paper is that there exists a critical value of $z$, say $z^*$, strictly between 0 and $q$ such that if $z$ is larger than $z^*$, then the two firms’ prices result in a region in which their products are independent of each other. That is, each firm’s demand depends only on its own price, not on the price of the other product. As a result, firm A behaves as if it is a monopolist supplying a product of quality level $z$, and firm B behaves as if it is a monopolist supplying a product of quality level $(q - z)$.

Interestingly enough, in this range of $z$, the two firms’ independently determined prices maximize their joint profits. That is to say, the well-known “double-markup” problem disappears completely, even though the two products are still complementary to a certain degree (i.e., $z^* < q$). Because the double mark-up problem vanishes, there is a region in which an increase in $z$ has a positive effect on firm B’s profit as well as on consumers’ surplus, thus yielding a win-win situation for both suppliers and consumers. This has implications for both cooperative R&D between firms and antitrust and intellectual property polices.
Given the dependence of the firms' profits on $z$, we study how firm A’s expansion in boundary affects firm B’s R&D incentive. We find that an increase in $z$ may have a positive effect on the other firm’s R&D incentive.

Related Literature

Brandenburger and Nalebuff (1996) point out the prevalence of "co-opetition," in which firms supplying complementary products compete with one another in dividing up industry profits. Bresnahan (1999) and Bresnahan and Greenstein (1999) discuss vertical competition as a salient feature of the computing industry. However, they do not model the phenomenon formally.

As Cournot pointed out in 1838, if a system consists of several components and each component is supplied by an independent monopoly, then we have the classic "complements problem." Each firm’s individual price is too high in terms of joint profit maximization.

There are several recent studies on complementarities between products, technologies and patents (Farrell and Katz 2000, Gilbert 2002, Lerner and Tirole 2002, Shapiro 2001). Farrell and Katz (2000) analyze the incentive of a monopolist in product A to enter product B’s market in order to force independent suppliers of B to charge lower prices, which increases its own profits made from product A. In a model of patent portfolios that allows a full range of complementarity and substitutability, Lerner and Tirole (2002) analyze the welfare effect of patent pools and evaluate several factors that encourage or hinder the formation of these pools.1

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1 Their model is closely related to our model in the sense that the degree of complementarity between patents is endogenously determined by licensing fees. In their model all users or licensees derive the same amount of marginal benefits from an additional patent. However, in our model since consumers of different types derive different amounts of marginal benefits from an additional product, some consumers buy only product A, while some others buy both products A and B. A second major difference is symmetry versus asymmetry. In their model all technologies are symmetric. In our model, products A and B are asymmetric. Because of the different assumptions, we can get the independency result if $z$ is sufficiently high.
Our paper has implications for the literature on tying. For instance, when MS expands the boundary of its operating system (O/S) by incorporating the web-browsing function, the effect of the integration depends on the quality of I.E. Our paper sheds light on how the quality level of (tied) I.E affects the prices of its own product and of its complementor’s product, their respective profits, and its complementor’s R&D effort.

Most of the vertical integration and tying articles mentioned above do not assume that consumers are heterogeneous with regards to quality. The assumption that consumers have heterogeneous valuation of quality plays a key role in our paper. Under this assumption, we get interesting results that are different from those in the existing literature, which emphasizes ‘price squeeze.’ If consumers were homogeneous in their valuation of quality, then our model would become very similar to that of Farrell and Katz (2000): an increase in z "price squeezes" product B and always has negative effects on firm B’s profit. With heterogeneous consumer preference, however, we show that an increase in z can have a positive effect on firm B’s profit and its R&D incentive. Also, the double mark-up problem disappears at a critical z which is strictly less than q.

Section 2 describes the model. Section 3 analyzes the pricing game and the effect of z on firms’ profits and social welfare. Section 4 discusses several related issues, and concluding remarks follow in the final section.

2. A Model of Product Boundaries

Consumers have different preferences with respect to products' quality. The intensity of their preferences is represented by θ. A type-θ consumer has per-period utility θQ+I, where I is her income spent on numeraire goods, and Q is a quality index of a product.

The distribution of consumer types on [0,1] is given by a cumulative distribution function G(θ) with continuous density g(θ). We define F(θ) as the proportion of

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consumers whose type is higher than \( \theta \) i.e., \( F(\theta) = 1 - G(\theta) \), and \( f(\theta) = -g(\theta) < 0 \). We make the standard assumption that the distribution of types satisfies the monotonic hazard rate condition: namely, \( -f(\theta)/F(\theta) \) is increasing in \( \theta \).\(^3\) This assumption ensures that a firm’s profit function is strictly quasi-concave and that the second-order condition for profit maximization is satisfied (see the Appendix).

There are two firms, A and B, that provide complementary products A and B, respectively. Product A provides some basic functions, and its performance level is measured by a parameter \( z \). Product B by itself does not provide any function, but enhances product A’s performance. The combination of products A and B (denoted by (A+B) hereafter) provides a higher performance level \( q \geq z \). The difference \((q - z)\) captures the extent to which product B enhances performance. Firm i’s (i=A, B) price and unit production cost are denoted by \( p_i \) and \( c_i \), respectively.

If \( z \) is equal to zero, then products A and B are strict complements. In contrast, if \( z = q \), then B would be completely superfluous. The relationship between products A and B depends on the level of \( z \). Thus, the parameter \( z \) can be interpreted as the boundary between products A and B.

Firms A and B set their prices sequentially. Firm A, the firm that produces the basic product without which product B is useless, sets its price first. Then, firm B sets its price. Given \( p_A \) and \( p_B \), consumers make their purchase decisions. In Section 4, we discuss the case in which firms A and B set their prices simultaneously.

We impose the following restrictions on key parameters throughout our analysis,

**Assumption 1:**

\[
q > c_A + c_B \\
q - z > c_B
\]

The first restriction implies that the maximum willingness to pay for product (A+B) exceeds its unit production cost. Without this restriction, (A+B) will never be supplied. The second restriction implies that the quality enhancement brought by product B exceeds \( c_B \). Without the second restriction, there will be no supply of product B. With

\(^3\) This monotonic hazard rate condition is satisfied by most widely used distributions.
the two restrictions, we focus on the more interesting case in which both firms A and B are active, a case in which the classic double mark-up problem may arise.

**Demand Systems for Products A and B**

Consumers have three options: to buy product A alone; to buy (A+B); and to buy neither. Clearly, their choices depend on their types as well as on the prices \( p_A \) and \( p_B \). The utility level of a type-\( \theta \) consumer is as follows:

- **Zero from no purchase**
  - \( V_A(\theta) = z\theta - p_A \) from buying product A
  - \( V_{A+B}(\theta) = q\theta - p_A - p_B \) from buying (A+B)

For a consumer to buy product A, we must have \( V_A(\theta) = z\theta - p_A \geq 0 \). That is to say, a necessary condition for a consumer to buy product A is \( \theta \geq \theta_A = p_A/z \). Similarly, a necessary condition for a consumer to buy (A+B) is that \( V_{A+B}(\theta) = q\theta - p_A - p_B \geq 0 \), or equivalently \( \theta \geq \theta_{A+B} = (p_A + p_B)/q \). Consumers get additional benefits of \((q-z)\theta\) by purchasing product B in addition to product A. Thus, a necessary condition for a consumer to buy product B in addition to product A is that \( \theta \geq \theta_B = p_B/(q-z) \).

Let us compare the relative sizes of the three cut-off points, \( \theta_A, \theta_B, \) and \( \theta_{A+B} \). \( V_A(\theta) \) and \( V_{A+B}(\theta) \) intersect with each other at one point, \( \theta_B \). Since the function \( V_{A+B}(\theta) \) has a steeper slope than that of \( V_A(\theta) \), there are only two possible cases: one case is \( \theta_A < \theta_{A+B} < \theta_B \); the other case is \( \theta_B < \theta_{A+B} < \theta_A \).

**Case 1. Virtually Independent Products:** \( \theta_A < \theta_{A+B} < \theta_B \).

This case is illustrated in Figure 1. Consumer types between \( \theta_A \) and \( \theta_B \) will buy only the basic product A and attain a quality level \( z \), whereas consumer types \( \theta \geq \theta_B \) will buy (A+B) and attain a quality level \( q \). That is, consumers with \( \theta \geq \theta_A \) will buy product A and consumers with \( \theta \geq \theta_B \) will additionally buy product B.
Thus, the demand functions for A and B are as follows:

\[
D_A(p_A, p_B) = F(\theta_A) = F\left(\frac{p_A}{z}\right) \quad (1)
\]

\[
D_B(p_A, p_B) = F(\theta_B) = F\left(\frac{p_B}{q-z}\right)
\]

In this case, demand for product A depends only on \(p_A\), and demand for product B depends only on \(p_B\). In other words, firms A and B may act as independent monopolies so long as their prices satisfy \(\theta_A < \theta_{A+B} < \theta_B\). We call this case “virtually independent products.”

Case 2: Virtually Strict Complements: \(\theta_B < \theta_{A+B} < \theta_A\)

Figure 2 illustrates this case. Consumers with \(\theta < \theta_{A+B}\) will buy neither products, but consumers with \(\theta \geq \theta_{A+B}\) will buy (A+B). None will buy product A alone.

Thus, the demand functions for products A and B are as follows:

\[
D_A(p_A, p_B) = D_B(p_A, p_B) = F\left(\frac{p_A + p_B}{q}\right) \quad (2)
\]

In this case, both the demand for product A and the demand for product B depend on the total price. Because product A provides a quality level \(z\) without the aid of product B, the two products are not strict complements in the usual sense of the term. However, as

\[4\] We can have the borderline case with \(\theta_B = \theta_{A+B} = \theta_A\).
demand system (2) shows, the market as a whole exhibits properties of strict complements. We call it the case of “virtually strict complements.” Figure 3 shows that the price space is divided into the two regions of virtually independent products and virtually strict complements.

[Figure 3 here]

3. The Pricing Game

This section analyses the pricing game between firms A and B and shows how the equilibrium is determined. Since firms A and B set their prices sequentially, by backward induction, we start with firm B’s optimal price given $p_A$.

3.1 Firm B’s Optimal Pricing

As demand systems (1) and (2) have shown, the demand function faced by firm B differs, depending on the relative size of $\theta_B$ and $\theta_A$. That is, if firm B sets $p_B$ such that $\theta_A < \theta_B$, then firm B faces demand (1), the case of virtually independent products. If firm B sets $p_B$ such that $\theta_A > \theta_B$, then firm B faces demand (2), the case of virtually strictly complementary products.

$$
D_B(p_A, p_B) = \begin{cases} 
D_{B1} = F\left(\frac{p_B}{q-z}\right) & \text{if} \ p_B \geq \frac{q-z}{z} p_A \\
D_{B2} = F\left(\frac{p_A + p_B}{q}\right) & \text{if} \ p_B \leq \frac{q-z}{z} p_A
\end{cases}
$$

Firm B’s demand curve consists of $D_{B1}$ and $D_{B2}$. $D_{B1}$ is independent of $p_A$. In contrast, $D_{B2}$ depends on the total price. Firm B’s demand curve is continuous, but has a kink at $p_B = \frac{q-z}{z} p_A$.\(^5\) Let $p_{B1}^*(z)$ and $p_{B2}^*(p_A)$ denote the unconstrained optimal price of $D_{B1}$ and $D_{B2}$, respectively. Since $D_{B1}$ is independent of $p_A$, $p_{B1}^*(z)$ depends on $z$, but not on $p_A$. In contrast, $p_{B2}^*(p_A)$ depends on $p_A$, but not on $z$.\(^6\)

\(^5\) At the kink, we have $\theta_A = \theta_B$.

\(^6\) As an illustration, in the case of a uniform distribution, $p_{B1}^* = \frac{q - z + c_B}{2}$, and $p_{B2}^* = \frac{q - p_A + c_B}{2}$. 
Figure 4 shows how $p_A$ affects firm B’s demand curve.

[Figure 4 here]

As $p_A$ increases, $D_{B2}$ shifts in. Thus if $p_A$ is high, firm B’s optimal price tends to be on $D_{B2}$; if $p_A$ is low, firm B’s optimal price tends to be on $D_{B1}$. The following lemma characterizes how firm B’s optimal price depends on $p_A$.

**Lemma 1** Firm B’s optimal price depends on $p_A$ and is continuous in $p_A$. There exist $p_{\underline{A}}$ and $p_{\overline{A}}$, where $0 < p_{\underline{A}} < p_{\overline{A}} < q-c_B$, such that

$$
p_B^* = \begin{cases} 
p_{B1}^*(z) & \text{if } p_A \leq p_{\underline{A}} \\
\frac{q-z}{z} p_A & \text{if } p_{\underline{A}} \leq p_A \leq p_{\overline{A}} \\
p_{B2}^*(p_A) & \text{if } p_{\overline{A}} \leq p_A \leq q - c_B.
\end{cases}
$$

**Proof:** See the Appendix.

For illustration, suppose that $p_A$ is zero. All consumers will get product A for free and will consider whether or not to buy product B additionally. Then firm B behaves as if it is a monopolist selling a product whose quality is $q-z$. Firm B maximizes its profit along $D_{B1}$ by setting its optimal price $p_{B1}^*(z)$, or by selling its products to consumers whose types are higher than $p_{B1}^*(z)/(q-z)$. As $p_A$ increases, fewer consumers will have product A, but as long as the lowest consumer type that buys product A is lower than $p_{B1}^*(z)/(q-z)$, firm B’s optimal price is not constrained by $p_A$. However, once $p_A$ exceeds $p_{\underline{A}}$, the constraint becomes binding, and we have $\theta_B=\theta_A$, and firm B’s optimal price

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7 As an illustration, in the case of a uniform distribution, $p_{\underline{A}} = \frac{z(q-z+c_A)}{2(q-z)}$ and $p_{\overline{A}} = \frac{z(q+c_A)}{2q-z}$.

8 Since the firm’s price determines which consumers will buy product B, setting its optimal price is equivalent to choosing its optimal cut-off point.
becomes $\frac{q-z}{z} p_A$. If $p_A$ is higher than $\bar{p}_A$, then firm B maximizes its profit along $D_{B2}$ by setting its price at $p_{B2}^*(p_A)$.

In summary, if $p_A < \underline{p}_A$, firm B maximizes its profit along $D_{B1}$, and we have $\theta_A < \theta_B$. If $\underline{p}_A \leq p_A \leq \bar{p}_A$, then we have $\theta_A = \theta_B$. If $\bar{p}_A < p_A$, then we have $\theta_A > \theta_B$.  

Figure 5 shows firm B’s best response to $p_A$ in the case of a uniform distribution. As Figure 5 shows, firm B’s best response is not monotonic.

[Figure 5 here]

### 3.2 Firm A’s Optimal Pricing

Taking firm B’s response into account, firm A’s reduced-form demand function is continuous\(^{10}\) and is as follows:

\[
D_A(p_A, \ p_{B2}^*(p_A)) = \begin{cases} 
D_{A1} = F\left(\frac{p_A}{\bar{p}_A}\right) & \text{if } p_A < \underline{p}_A \\
D_{A1} = F\left(\frac{p_A}{\bar{p}_A}\right) & \text{if } \underline{p}_A \leq p_A \leq \bar{p}_A \\
D_{A2} = F\left(\frac{p_A + F(p_A)q}{q}\right) & \text{if } \bar{p}_A \leq p_A
\end{cases}
\]

If $p_A$ is set below $\underline{p}_A$, as Lemma 1 shows, products A and B become virtually independent, and firm A’s demand is $F\left(\frac{p_A}{\bar{p}_A}\right)$. If $p_A$ lies between $\underline{p}_A$ and $\bar{p}_A$, we have $\theta_A = \theta_B$. However, after substituting $p_{B2}^* = \frac{q-z}{z} p_A$ into equation (2), the resulting demand for firm A is $D_{A1} = F\left(\frac{p_A}{\bar{p}_A}\right)$. That is, firm A perceives its product as independent of product B, while in fact products A and B are consumed together (called the case of "pseudo

\(^{9}\) If $p_A > (q - c_B)$ then firm B’s best response is a corner solution $c_B$, but there is no demand for either product because the total price of (A+B) exceeds the consumers’ maximum willingness to pay.

\(^{10}\) Firm A’s demand is continuous in $p_A$: If $p_A = \bar{p}_A$, then $\frac{p_A}{\bar{p}_A} = \frac{1}{q} \left[ \bar{p}_A + p_{B2}^*(\bar{p}_A) \right]$. 

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complements" in section 3.3). If firm A sets \( p_A \) above \( \bar{p}_A \), then products A and B become virtually strict complements, and firm A’s demand becomes \( D_{A2} \).

Let us define profit functions \( \Pi_{A1} \) and \( \Pi_{A2} \) corresponding to \( \Pi_{A1} \) and \( \Pi_{A2} \), respectively. That is, \( \Pi_{A1}(p_A; z) = F\left(\frac{p_A}{z}\right)(p_A - c_A) \), and \( \Pi_{A2}(p_A) = F\left(\frac{p_A + p_A^*(p_A)}{q}\right)(p_A - c_A) \). Firm A maximizes \( \Pi_{A1}(p_A; z) \) subject to the constraint \( p_A \leq \bar{p}_A \) and maximizes \( \Pi_{A2}(p_A) \) subject to the constraint \( p_A \geq \bar{p}_A \). Let \( p_{A1}^* \) and \( p_{A2}^* \) denote the unconstrained optimal prices for \( \Pi_{A1} \) and \( \Pi_{A2} \), respectively. It is clear that \( p_{A1}^* \) depends on \( z \) (i.e., \( p_{A1}^*(z) \)), but \( p_{A2}^* \) does not.\(^{11}\)

The profit function \( \Pi_{A1} \) increases in \( z \), while the profit function \( \Pi_{A2} \) is independent of \( z \). If \( z \) is sufficiently large, then \( \Pi_{A1} \) dominates \( \Pi_{A2} \). That is to say, if \( z \) is large, the globally optimal price for firm A is determined by \( \Pi_{A1} \). However, if \( z \) is small, it is determined by \( \Pi_{A2} \). Lemma 2 below shows that either \( p_{A1}^*(z) \) or \( p_{A2}^* \) will be firm A’s optimal price, depending on the level of \( z \).

**Lemma 2:** Firm A’s optimal price is as follows:

\[
p_{A}^{*} = \begin{cases} 
    p_{A2}^* & \text{if } z \leq z^+ \\
    p_{A1}^*(z) & \text{if } z \geq z^+, 
\end{cases}
\]

where \( z^+ \) is defined by \( \Pi_{A1}(p_{A1}^*; z^+) = \Pi_{A2}(p_{A2}^*) \), and \( 0 < z^+ < q - c_B \). At \( z^+ \), firm A’s optimal price is either \( p_{A2}^* \) or \( p_{A1}^*(z^+) \). For \( z \leq z^+ \), \( p_{A2}^* > \bar{p}_A \), and for \( z \geq z^+ \), \( p_{A1}^*(z) < \bar{p}_A \), which implies that \( p_{A1}^*(z^+) < p_{A2}^* \).

**Proof:** See the Appendix.

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\(^{11}\) In the case of a uniform distribution, \( p_{A1}^*(z) = \frac{z + c_A}{2} \), and \( p_{A2}^* = \frac{q - c_B + c_A}{2} \).
If \( z \) is larger than or equal to \( z^+ \), firm A maximizes its profit by choosing \( p_{A1}^*(z) \), which is the unconstrained optimal price for \( \text{DA1} \). Since \( p_{A1}^*(z) < \bar{p}_A \), if \( z \geq z^+ \), then firm A behaves as if it maximizes its profit along \( \text{DA1} \) without the constraint \( p_A \leq \bar{p}_A \).

If \( z \) is smaller than or equal to \( z^+ \), firm A maximizes its profit by choosing \( p_{A2}^* \), which is the unconstrained optimal price for \( \text{DA2} \). Since \( p_{A2}^* > \bar{p}_A \), if \( z \leq z^+ \), then firm A behaves as if it maximizes its profit along \( \text{DA2} \) without the constraint \( p_A \geq \bar{p}_A \).

### 3.3 Product Relationship

This section analyses how the product relationship depends on \( z \).

Suppose that \( z \) is smaller than or equal to \( z^+ \). From Lemma 2, we know that firm A sets its price at \( p_{A2}^* \), which is higher than \( \bar{p}_A \). Since \( p_{A2}^* > \bar{p}_A \), we have \( \theta_B < \theta_A \), the case of virtually strict complements. Even though \( z \) is positive, the two products exhibit the characteristics of perfectly complementary products.

Suppose that \( z \) is larger than or equal to \( z^+ \). Then, firm A behaves as if it is a monopolist selling product of quality \( q \). Its optimal price is \( p_{A1}^*(z) \), which is less than \( \bar{p}_A \).

In order to determine the product relationship, we need to compare \( p_{A1}^*(z) \) and \( \bar{p}_A \). The following lemma shows that \( p_{A1}^*(z) \leq \bar{p}_A \) if and only if \( z \geq \frac{q \cdot c_A}{c_A + c_B} \).

**Lemma 3:** We have \( p_{A1}^*(z) \leq \bar{p}_A \) if and only if \( z \geq \frac{q \cdot c_A}{c_A + c_B} \).

**Proof:** See the Appendix.

Thus, if \( z^+ \leq z \leq \frac{q \cdot c_A}{c_A + c_B} \), then \( p_A \leq p_{A1}^*(z) < \bar{p}_A \), and thus firm B sets its price \( p_B^* = \frac{q + z}{z} \cdot p_{A1}^* \). In this case, we have \( \theta_B = \theta_A \). In equilibrium, products A and B are consumed together. While firm A sets its price independently, firm B’s optimal cut-off point is constrained by firm A’s pricing, and firm B behaves as if the two goods are strictly
complementary. We call this case pseudo complementarity. Later, we show that several interesting features occur in this case.

If \( z \geq \max\{z^+, \frac{q c_B}{c_A + c_B}\} \), then \( p^*_A (z) \leq p^*_B \), and thus firm B also behaves as if it is a monopolist selling a product of quality \((q-z)\). It sets its price such that \( \theta_B > \theta_A \), and we have the case of virtual independency. Firms behave as if their products are independent of each other.

In summary, if \( \frac{q c_B}{c_A + c_B} < z^+ \), we have two regimes: for \( z \leq z^+ \), we have the case of virtually complementary products; for \( z \geq z^+ \), we have the case of virtually independent products. If \( z^+ < \frac{q c_A}{c_A + c_B} \), we have three regimes: for \( z \leq z^+ \), we have the case of virtually complementary products; for \( z^+ \leq z \leq \frac{q c_A}{c_A + c_B} \), we have the case of pseudo complements; for \( z \geq \frac{q c_A}{c_A + c_B} \), we have the case of virtually independent products. Figure 6 illustrates these cases.

[Figure 6 here]

**Proposition 1.** Let \( z^* \) denote \( \max\{z^+, \frac{q c_B}{c_A + c_B}\} \). The relationship between products A and B depends on \( z \) in the following way:

(a) If \( z \leq z^+ \), products A and B are virtually strict complements.
(b) If \( z^+ \leq z \leq \frac{q c_A}{c_A + c_B} \), products A and B are pseudo complements.
(c) If \( z \geq z^* \), products A and B are virtually independent.

**3. 4 Double Mark-up Problem**

Suppose that a vertically integrated monopolist supplies both products A and B. This firm maximizes its total profits from the sales of these products by choosing two cut-off points, \( x_A \) and \( x_B \), where types higher than \( x_A \) buy product A, and types higher
than $x_B$ buy product B additionally. The profit function can be written in terms of $x_B$ and $x_A$ with the constraint that $x_A \leq x_B$.

$$\Pi^m = F(x_A)(p_A - c_A) + F(x_B)(p_B - c_B)$$

$$= F(x_A)(z x_A - c_A) + F(x_B)((q-z) x_B - c_B)$$

$$= zF(x_A)(x_A - \frac{c_A}{z}) + (q-z)F(x_B)( x_B - \frac{c_B}{q-z})$$

The profit function consists of two parts, $zF(x_A)(x_A - \frac{c_A}{z})$ and $(q-z)F(x_B)( x_B - \frac{c_B}{q-z})$. The first one is equivalent to the profit function of a monopolist whose product quality is $z$ and marginal cost is $\frac{c_A}{z}$, and the second one is equivalent to the profit function of a monopolist whose product quality is $(q-z)$ and marginal cost is $\frac{c_B}{q-z}$.

Suppose that the firm maximizes its profit piecewise without the constraint that $x_A \leq x_B$. Then, the firm’s optimal $x_A$ depends on only $\frac{c_A}{z}$, and the firm’s optimal $x_B$ depends on only $\frac{c_B}{q-z}$. If $\frac{c_A}{z} \leq \frac{c_B}{q-z}$, or equivalently if $z \geq \frac{q c_A}{c_A + c_B}$, (since a firm’s optimal cut-off point increases in its marginal cost), then we have $x_A^* \leq x_B^*$. That is, we can ignore the constraint that $x_A \leq x_B$. Without the constraint that $x_A \leq x_B$, the two parts of the profit function are decoupled. In other words, the firm behaves as if these two products are completely independent.

If $\frac{c_A}{z} > \frac{c_B}{q-z}$, or equivalently if $z < \frac{q c_A}{c_A + c_B}$, then the constraint is binding, and we have $x_B^* = x_A^*$. The profit function becomes $F(x_{A+B})(p_A + p_B - c_A - c_B)$, where $x_{A+B} = \frac{p_A + p_B}{q}$.

The integrated firm’s profit depends on the total price, not the level of $z$. That is, the firm behaves as if these two products are strict complements of one another.

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12 The firm can be regarded as maximizing its profit by selling product A and product (A+B). See Johnson and Myatt (2002) for multiproduct quality competition. Also, Ellison (2002) analyzes add-on pricing (i.e., the prices of higher quality products are not advertised, while the prices of the basic products are advertised).
Lemma 4. If $z < \frac{c_A}{c_A + c_B}$, the vertically integrated monopolist behaves as if these two products are strict complements of one another. If $z \geq \frac{c_A}{c_A + c_B}$, the firm behaves as if these two products are completely independent.

Let us return to the case in which firms A and B set their prices non-cooperatively. If $z \geq z^*$, we have the case of virtual independence. The sum of $\Pi_A$ and $\Pi_B$ is exactly the same as the profit function of the integrated monopolist. Thus, even though firm A and firm B sets their prices non-cooperatively, their prices maximize the industry profit.

Proposition 2: If $z \geq z^*$, even though firms A and B set their prices non-cooperatively, their prices are the same as those set by a vertically-integrated monopolist. Thus, the double mark-up problem disappears.

3.5 The Effect of $z$ on firms’ prices and profits.

So far, we have analysed the market outcome given $z$. This section does the comparative statics analysis. We assume that the unit production cost of product A does not change with $z$, which is a good assumption for the software industry. For instance, even though MS, incurring R&D costs, expands the functions of MS Windows, the unit production cost of MS Windows remains the same as before.

If we allow $c_A$ to increase with $z$, the main comparative statics results still hold as long as $\frac{z}{c_A(z)}$ is increasing in $z$. If $\frac{z}{c_A(z)}$ is increasing in $z$, then as Figure 6 shows, the product relationship changes monotonically in $z$, from complementary goods to independent goods, or from complementary goods, to pseudo complementary goods, and to independent goods. Also, the results in Propositions 3 and 4 are still valid for the cases of pseudo complements and virtual independence. The only changes occur in the case of virtually strict complements. Since $c_A(z)$ increases with $z$, in the case of virtually strict complements, firm A’s price increases in $z$; firm B’s price decreases in $z$; firm A’s profit decreases in $z$; firm B’s profit decreases in $z$. 

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13 If we allow $c_A$ to increase with $z$, the main comparative statics results still hold as long as $\frac{z}{c_A(z)}$ is increasing in $z$. If $\frac{z}{c_A(z)}$ is increasing in $z$, then as Figure 6 shows, the product relationship changes monotonically in $z$, from complementary goods to independent goods, or from complementary goods, to pseudo complementary goods, and to independent goods. Also, the results in Propositions 3 and 4 are still valid for the cases of pseudo complements and virtual independence. The only changes occur in the case of virtually strict complements. Since $c_A(z)$ increases with $z$, in the case of virtually strict complements, firm A’s price increases in $z$; firm B’s price decreases in $z$; firm A’s profit decreases in $z$; firm B’s profit decreases in $z$. 

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Let us characterize how $z$ affects $p_A^*$, $p_B^*$, and the total price ($p_A^* + p_B^*$). For $z \in [0, z^+)$, the market outcome is the same as that of $z = 0$, and prices do not depend on $z$.

For all $z \geq z^+$, firm $A$ chooses its cut-off point as if it is an independent monopolist. As Lemma 2 shows, since $p_{A1}^*(z^+) < p_{A2}^*$, there is a downward jump in firm A’s optimal price at $z^+$. That is, as $z$ exceeds $z^+$, the price of product $A$ actually falls.

The intuition is as follows. Product $A$ and product $(A+B)$ are substitutes for one another. In order to induce consumers to buy product $A$ instead of product $(A+B)$, firm $A$ must lower its price relative to $(p_A + p_B)$. If $z$ is small, it is not profitable for firm $A$ to do so. Instead, firm $A$ sells product $A$ as a part of product $(A+B)$, not product $A$ alone. In this case, firm $A$ charges a high price to induce low $p_B$. However, if $z$ is large, it becomes profitable for firm $A$ to induce some consumers to buy product $A$ alone.

For all $z \geq z^+$, firm A’s profit function can be written in terms of the cut-off point $\theta_A$.

$$
\Pi_A (\theta_A) = F(\theta_A) (z\theta_A - c_A)
$$

It is equivalent to the profit maximization problem of a monopolist selling a good with a unit production cost of $c_A/z$. (Note that $\theta_A^*$ maximizes $F(\theta_A)(\theta_A - c_A/z)$ as well as $F(\theta_A)(z\theta_A - c_A)$. As $z$ increases, the unit production cost $c_A/z$ becomes smaller, and the firm’s optimal cut-off point $\theta_A$ becomes smaller. As $z$ increases, more consumers buy product $A$.

For $z \geq z^+$, firm A’s optimal price satisfies the following first-order condition:

$$
p_A^* = c_A - z \frac{F(\theta_A)}{f(\theta_A)}
$$

Since $- \frac{F(\theta_A)}{f(\theta_A)}$ is decreasing in $\theta_A$ and $\theta_A$ is decreasing in $z$, the right-hand side is increasing in $z$. Thus, $p_{A1}^*(z)$ increases in $z$ beyond $z^+$. As an illustration, Figure 7 shows firm A’s optimal price as a function of $z$ in the case of a uniform distribution.

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14 Firm A’s perceived reduced-form demand curve, taking into account firm B’s optimal response, becomes $F\left(\frac{p_A}{z^*}\right)$ at $z^+$. The downward jump in $p_A^*$ occurs at $z^+$, not $z^*$. 

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16
For $z \in [z^+, \frac{q - c_A}{q - z}]$, we have the case of pseudo complements: firm B’s optimal cut-off point is constrained by $\theta_A$ so that $\theta_B = \theta_A$. In this case, as $z$ increases, firm A lowers its optimal cutoff point, allowing firm B to sell its products to additional consumers. Since more customers purchase products A and B, the total price ($p_A^* + p_B^*$) must have decreased, which implies that $p_B^*$ decreases in $z$ for $z \in [z^+, \frac{q - c_A}{q - z}]$ and that the decrease in $p_B^*$ more than offsets the increase in $p_A^*$.

Beyond $z^*$, firm B, like firm A, behaves as if it is an independent monopolist supplying a product of quality $(q - z)$. Firm B’s profit maximization problem can also be regarded as choosing its cut-off point $\theta_B$.

$$\Pi_B (\theta_B) = (q - z)F(\theta_B) (\theta_B - \frac{c_B}{q - z})$$

It is equivalent to the profit maximization problem of a monopolist selling a product with a unit production cost of $\frac{c_B}{q - z}$. As $z$ increases, its optimal cut-off point increases, implying that fewer consumers buy product B. Beyond $z^*$, an increase in $z$ pushes down $\theta_A$ and pushes up $\theta_B$, shifting customers from product (A + B) to product A alone.

Beyond $z^*$, firm B’s optimal price satisfies the following first-order condition:

$$p_B^* = c_B - \frac{F(\theta_B)}{f(\theta_B)} (q - z)$$

Since $\frac{F(\theta_B)}{f(\theta_B)}$ is decreasing in $\theta_B$ and $\theta_B$ is decreasing in $q - z$, the right-hand side is increasing in $q - z$. Thus, $p_B^*(z)$ decreases as $z$ increases. In the case of a uniform distribution, the increase in $p_A^*$ is exactly offset by a decrease in $p_B^*$, making the total price constant.\footnote{Other distributions exist under which the sum of the two prices is constant. For instance, $g(\theta) = 40$ for $0 \leq \theta \leq 1/2$ and $g(\theta) = 4-4\theta$ for $1/2 \leq \theta \leq 1$.}
**Proposition 3:**
(a) If products A and B are virtually strict complements, z does not affect firms’ prices.
(b) If products A and B are pseudo complements, then \( p_A^* \) is \( p_{A1}^*(z) \). Since \( p_{A1}^*(z^+) < p_{A2}^* \), there is a downward jump in firm A’s optimal price at \( z^+ \). \( p_{A1}^*(z) \) increases in z, and \( p_B^* \) is \( \frac{q-z}{z} p_A \), which decreases in z. As z increases, the total price decreases and more consumers buy product (A+B).
(c) If products A and B are virtually independent, then \( p_A^* \) is \( p_{A1}^*(z) \), which increases with z, and \( p_B^* \) is \( p_{B1}^*(z) \), which decreases as z increases. As z increases, more consumers buy product A despite a price increase in \( p_A^* \), and fewer consumers buy product B despite a price decrease in \( p_B^* \).

Let us look at how the firms’ profits change with z. For \( z \in [0, z^+) \), the market outcome is the same as that of \( z = 0 \). For \( z > z^+ \), firm A’s profit increases with z, and firm B’s profit decreases as z increases for \( z > z^+ \). As Proposition 3 shows, \( p_A^* \) jumps down at \( z^+ \). Thus firm B’s profit jumps up at \( z^+ \). As an illustration, Figure 8 shows firm A’s and B’s profits as functions of z in the case of a uniform distribution.

[Figure 8 here]

Interestingly, for some range of z, firm B’s profit is higher than that of \( z = 0 \). For instance, if \( q = 1 \), \( c_A = c_B = 0 \) and the density function \( g(\theta) \) is uniform, then firm B’s profit for a positive z is higher than that for \( z = 0 \) as long as z is between \( \frac{1}{2} \) and \( \frac{3}{4} \). That is, when firm A increases the level of z to the range \([\frac{1}{2}, \frac{3}{4}]\), it actually benefits firm B.

**Proposition 4:** The firms’ profits are independent of z for \([0, z^+)\). Beyond \( z^+ \), firm A’s profit increases continuously with z, whereas firm B’s profit jumps up at \( z^+ \), but decreases with z afterwards. Thus, over a certain range, an increase in z can actually benefit firm B.
**Proof** See the Appendix.

Let us analyse how $z$ affects the industry profits, which is the sum of profits by firms A and B. In the case of virtually strict complements, $z$ does not affect the firm A’s and B’s profits.

In the case of pseudo complements, $\theta_A = \theta_B$ and consumers buy only product $(A + B)$, so the double mark-up problem persists: the total price determined non-cooperatively by the two independent firms is too high in terms of the joint profits. Nevertheless, we can show that as $z$ increases, the total price becomes lower and closer to the vertically integrated monopolist’s optimal price, and thus their total profits increase.

In the case of virtually independent products, Proposition 2 states that the sum of the firms’ profits without cooperation is the same as the vertically integrated monopolist’s profits, which increases in $z$. Consequently, the sum of the two firms’ profits increase in $z$ for $z \geq z^+$.

**Proposition 5:** The industry’s total profits do not depend on $z$ for $z < z^+$, but increase in $z$ for $z \geq z^+$.

**Proof.** See the Appendix.

### 3.6 Welfare Analysis

This section analyses how an increase in $z$ affects consumer surplus and social welfare. Note that if $z$ is smaller than $z^+$, $z$ does not affect the firms’ prices, profits or consumer surplus.

In the case of pseudo complements, each firm’s price is a function of $z$. Thus, consumer surplus is a function of $z$. As Proposition 3 shows, the total price of product $(A + B)$ decreases in $z$, and, as a result, consumer surplus increases in $z$. From Proposition 5, we know that the firms' total profits increase in $z$, too. Thus, social welfare is unambiguously increasing in $z$ in the case of pseudo complements.
In the case of virtual independence, $\theta_B$ increases in $z$, but $\theta_A$ decreases in $z$. Thus, some new consumers buy product A, and some existing consumers of product $(A + B)$ switch to product A. The total consumer surplus is as follows:

$$CS(z) = \int_{\theta_A(z)}^{\theta_B(z)} (zs - p_A(z))dG(s) + \int_{\theta_A(z)}^{\theta_B(z)} (qs - p_A(z) - p_B(z))dG(s).$$

The welfare effect of $z$ on consumer surplus is ambiguous. However, if $G(\theta)$ is a uniform distribution, the price of product $(A+B)$ does not change with $z$. Since the price of product $(A+B)$ remains unchanged, by revealed preference, consumers are better off when they switch from product $(A+B)$ to product A. Also, in light of Proposition 5 and the fact that $z^{*}\geq z^+$, the firms' total profits increase in $z$ for $z \geq z^*$. Thus, social welfare unambiguously increases in $z$ under a uniform distribution for $z \geq z^*$.

**Proposition 6**: In the case of pseudo complements, social welfare increases in $z$. In the case of virtual independence, the effect of an increase in $z$ on social welfare is ambiguous. However, in the case of a uniform distribution, social welfare increases in $z$.

Suppose that firm A can increase $z$ by undertaking a costly R&D project. Let us compare firm A's optimal $z$ and the socially optimal $z$. To understand the differences between the two, let us identify two effects: a “Business Stealing Effect” and “Spence’s quality setting effect.” First, firm A increases its profit at the expense of firm B’s profit for $z>z^+$. However, firm A does not internalise the negative effect of $z$ on firm B’s profit, which makes firm A’s optimal $z$ larger than the socially optimal one. Second, when firm A increases $z$, it has a positive effect on consumer surplus since some consumers use only product A, but firm A fails to appropriate the increase in consumer surplus. Thus, the second effect makes firm A’s optimal $z$ smaller than the socially optimal one. The

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16 In the case of a uniform distribution, firm A sets its price at $\frac{z + c_A}{2}$ and firm B sets its price at $\frac{q - z + c_B}{2}$. The total price is $\frac{q + c_A + c_B}{2}$, which is independent of $z$.

17 As pointed out by Spence, when a firm sets its quality level, it cares about only the effect of the quality setting on the marginal type, not an average type.
relative size of these two opposing effects is generally ambiguous. However, as section 4.2 shows, if \( c_A = c_B = 0 \), then we can get a clearer welfare result.

4. Extensions and Discussions

4.1. The Effect of \( z \) on Firm B’s Innovation.

We now analyse how \( z \) affects firm B’s incentive to increase \( q \) by analysing the sign of \( \frac{\partial^2 \pi_B}{\partial z \partial q} \). Not unexpectedly, the effect of \( z \) on firm B’s R&D incentive depends on the product relationship between A and B.

**Proposition 7:** If the two products are virtually strict complements, then \( z \) has no effect on firm B’s incentive to increase \( q \). If the two products are pseudo complements, then an increase in \( z \) strengthens firm B’s incentive to increase \( q \). If the two products are virtually independent, then an increase in \( z \) weakens firm B’s incentive to increase \( q \).

**Proof.** See the Appendix.

If the two products are virtually strict complements, \( z \) has no effect on the firms’ profits or on firm B’s incentive to increase \( q \). If the two products are pseudo complements, firm B’s optimal cut-off point is constrained by \( \theta_A^* \), and all consumers buy product (A +B). As \( z \) increases, firm A sells its product to more consumers, implying that firm B also can sell more quantities of its product. Since firm B sells its product to more consumers, its marginal profit with respect to \( q \) is higher, i.e., \( \frac{\partial^2 \pi_B}{\partial z \partial q} \) is positive.

If the two products are virtually independent, some patrons buy product A, while other patrons buy product (A+B). Product (A+B) and product A are substitutes for one another. As \( z \) increases, product A becomes a more attractive substitute for product (A+B), making the positive impact of \( q \) on firm B’s demand smaller. Thus, an increase in \( z \) lowers firm B’s marginal return to \( q \), i.e., \( \frac{\partial^2 \pi_B}{\partial z \partial q} \) is negative.
4.2. Special Case of Zero Unit Production Cost

If $c_A$ and $c_B$ are zero, a good approximation for software industries, we obtain clearer social welfare results. In this case, if $z \leq z^+$, then products A and B are strict complements; if $z \geq z^+$, products A and B are virtually independent.\(^{18}\)

For $z \geq z^+$, firm A maximizes $zF(\theta_A)\theta_A$ by choosing $\theta_A$, and firm B maximizes $(q-z)F(\theta_B)\theta_B$ by choosing $\theta_B$. The optimal $\theta_A^*$ satisfies $F(\theta_A^*) + \theta_A^* f(\theta_A^*) = 0$. For all $z \geq z^+$, firm A’s optimal $\theta_A^*$ is the same as firm B’s optimal $\theta_B^*$.\(^{19}\) That is, even though the two products become independent of each other, we have $\theta_A^* = \theta_B^*$, and consumers buy both products. Firm A’s optimal price becomes $z \theta_A^*$, and firm B’s optimal price becomes $(q-z) \theta_B^*$, and the total price is $q \theta_A^*$. Note that beyond $z^+$, $z$ affects neither the firms’ optimal cut-off point nor the total price.

At $z^+$, the total price jumps down. However, beyond $z^+$, $z$ has no effect on the total price, on the number of consumers buying the two products, or on social welfare. For $z > z^+$, the only effects of $z$ are changing the firms’ prices as well as the division of the industry’s total profits between firms A and B.

**Proposition 8:** If $c_A = c_B = 0$, then for $z > z^+$, the two products become independent of each other; consumers buy both products; and $z$ does not affect the number of consumers buying the two products.

Let us suppose that firm A undertakes a costly R&D process to increase $z$. Since firm A’s profit is independent of $z$ for $z < z^+$, firm A will contemplate R&D only to increase $z$ above $z^+$. However, since $z$ does not affect the number of consumers buying the two products beyond $z^+$, setting $z$ larger than $z^+$ would be socially wasteful. Thus, after $z^+$, firm A’s R&D incentive is too strong as compared with the social optimum.

However, the total price jumps down at $z^+$, and firm B’s profit may be higher form some

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\(^{18}\) If $c_A$ and $c_B$ are zero, then the region of pseudo complements disappears.

\(^{19}\) The optimal $\theta_B^*$ satisfies $F(\theta_B^*) + \theta_B^* f(\theta_B^*) = 0$. 
Proposition 9: In the case of $c_A = c_B = 0$, increasing $z$ beyond $z^+$ is socially wasteful. However, increasing $z$ could improve social welfare as compared to the case of $z = 0$.

4.3. Simultaneous Moves Game

So far we have assumed that firm A sets its price before firm B does. If instead firms A and B set their prices simultaneously, then it can be shown in the case of uniform distribution that our results are robust as long as a pure Nash equilibrium exists. More specifically, there exist two critical values of $z$, $z_{\min}$, $z_{\max}$, with $z_{\min} < z_{\max}$: for $z < z_{\min}$, the Nash equilibrium yields virtually strict complements; for $z > z_{\max}$, the Nash equilibrium yields virtually independent products; and for $z_{\min} < z < z_{\max}$, no pure strategy Nash equilibrium exists. In other words, as long as the pure strategy Nash equilibrium exists, our results are robust.

4.4. Alternative Demand Specification

Let us check the robustness of our results under a different demand specification. We assume that product A provides an independent stand-alone basic function, the value of which is $\nu$ to all types of consumers. For instance, Microsoft O/S provides stand-alone benefits even without including I. E. After product A expands its boundary by incorporating a new function $z$, consumers of type $\theta$ get utility of $z\theta + \nu$ from product A, whereas they get utility of $q\theta + \nu$ from product (A+B). Since $\nu$ is a constant for all consumers who buy A or (A+B), if $p_A$ is replaced by $(p_A - \nu)$, then the analysis under this new demand specification would be identical to our analysis in Sections 2-3. In other words, the results obtained in this paper are robust with respect to this alternative demand specification. It would be worthwhile to study further whether our results are robust under more generalized demand specifications featuring a positive correlation in the valuation between the two products.
4.5. Unbundling Product A into Two Separate Products

Let us consider the possibility of unbundling product A with $z > 0$ into two separate products: product A without $z$ as one product and a variant of B (call it B1) as a second product. The combination of products A and B1 provides a quality level $z$, while the combination of products A and B provides a higher quality level $q$. That is, product A is strictly complementary to B or B1, and product B1 competes directly with product B. It can be shown that, if firm A sets $p_A$ first, and, subsequently, firms A and B set prices for B1 and B simultaneously, then unbundling will have no effect on our results because the equilibrium price of B1 is zero. However, the implications of changing the sequence of moves requires a full-blown analysis, and the topic seems worthwhile to pursue in the future.

5. Conclusion

A dominant firm in one layer of a multi-layered system often seeks to extend the functions of its products to include functions that are traditionally covered by firms in other layers. The definition of product boundaries changes continuously as a consequence of vertical competition. In this paper, we solve for the equilibrium prices and profits of firms A and B for different boundary values $z$. We discover that the relationship between products A and B varies with $z$ and that there are three possible relationships: virtually strict complements, pseudo complements, and virtually independent products. Furthermore, if $z$ exceeds a certain threshold value, the well-known "double mark-up" problem vanishes, even though one product is still strictly complementary to the other product. The leading firm’s expansion of product boundary not only increases its own profits, but also may increase the other firm’s profits and social welfare. Thus, there is a region in which the profits of both firms, as well as consumer surplus, increase with $z$.

An implication is that firm B might have an incentive to share its intellectual property rights with firm A to shift the product boundary and to eliminate the double mark-up problem. This suggests a potentially new perspective on the DOJ intellectual property guidelines and a new direction for case studies.
One direction of extension of this paper would be to explicitly model the equilibrium product boundary when both firms engage in a battle over their product boundary. A second direction would be to model efforts by leading firms to chart the direction of future technologies through the introduction of enabling technologies and controlling interfaces.

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Appendix

Proof that the profit function is strictly quasi-concave under the monotonic hazard rate condition.

Consider a monopolist supplying a product of quality level $z$. Without any loss of generality we assume that the marginal cost of the product is zero. $p$ denotes the price of the product, and $\theta$ denotes the firm’s cutoff point i.e., the lowest consumer type buying the product. By definition, $\theta = p/z$. Since there is a one-to-one mapping between $p$ and $\theta$, the firm maximizes $z\theta F(\theta)$ by choosing its optimal cutoff point $\theta$. We have the following first-order condition:

$$zF(\theta)+z\theta f(\theta) = 0$$

The second derivative of the profit function with respect to $\theta$ is $2zf(\theta)+z\theta f'(\theta)$. By the first-order condition, namely $\theta = -F(\theta)/f(\theta)$, the second derivative becomes

$$2zf(\theta)-z\theta f'(\theta)/f(\theta)$$

The monotonic hazard rate condition (i.e., $-f/F$ is increasing) is equivalent to that $(f(\theta))^2-F(\theta)(f'(\theta))>0$. Since $f(\theta)<0$, the condition implies that $zf(\theta)-z\theta f'(\theta)/f(\theta)<0$. Since $f(\theta)<0$, we have $2zf(\theta)-z\theta f'(\theta)/f(\theta)<0$. That is, under the monotonic hazard rate condition the second-order condition is satisfied. Since the profit function is locally strictly concave in $\theta$ whenever its slope is zero, the function is strictly quasi-concave in $\theta$. Q.E.D

Proof of Lemma 1

Firm B’s profit function is composed of the two underlying profit functions. It maximizes $\Pi_{B1} = F(\frac{pB}{q-z})(pB-c_B)$ subject to $p_B \geq \frac{q-z}{z} p_A$ but maximizes $\Pi_{B2} = F(\frac{pA+pB}{q}) (p_B-c_B)$ subject to $p_B \leq \frac{q-z}{z} p_A$. The two profit functions intersect at $p_B = \frac{q-z}{z} p_A$, i.e.,

$$\frac{1}{q-z} \left( \frac{q-z}{z} p_A \right) = \frac{1}{q} \left( p_A + \frac{q-z}{z} p_A \right).$$
Under the monotonic hazard rate condition, each of these two functions is strictly quasi-concave in $p_B$ and has a single peak. Let $p_{B1}^*$ and $p_{B2}^*$ denote the unconstrained optimal $p_B$ for the profit functions, $\Pi_B(p_B)$ and $\Pi_B(p_B)$, respectively. The first-order condition of these profit functions are as follows:

\[
\frac{\partial \Pi_{B1}}{\partial p_B} = F\left( \frac{p_B}{q-z} \right) + f\left( \frac{p_B}{q-z} \right) \frac{1}{q-z} (p_B-c_B) = 0
\]

\[
\frac{\partial \Pi_{B2}}{\partial p_B} = F\left( \frac{p_A+p_B}{q} \right) + f\left( \frac{p_A+p_B}{q} \right) \frac{1}{q} (p_B-c_B) = 0
\]

Let us evaluate these two first-order conditions at the kink $p_B = \frac{q-z}{z} p_A$.

\[
\frac{\partial \Pi_{B1}}{\partial p_B} \bigg|_{z} \frac{q-z}{z} p_A = F\left( \frac{p_A}{z} \right) + f\left( \frac{p_A}{z} \right) \frac{1}{z} (p_A-c_B) = 0 \quad (A1)
\]

\[
\frac{\partial \Pi_{B2}}{\partial p_B} \bigg|_{z} \frac{q-z}{z} p_A = F\left( \frac{p_A}{z} \right) + f\left( \frac{p_A}{z} \right) \frac{1}{z} (p_A-c_B) = 0 \quad (A2)
\]

(A1) and (A2) can be written in the following ways.

\[
p_A + \frac{F\left( \frac{p_A}{z} \right)}{f\left( \frac{p_A}{z} \right)} \frac{z}{q-z} = \frac{z}{q-z} c_B \quad (A1)'
\]

\[
p_A + \frac{F\left( \frac{p_A}{z} \right)}{f\left( \frac{p_A}{z} \right)} \frac{qz}{q-z} = \frac{z}{q-z} c_B \quad (A2)'
\]

Since $F(\theta)/f(\theta)$ is increasing in $\theta$, the left-hand side of (A1)’ and (A2)’ is monotonically increasing in $p_A$. Thus, given $q$, $z$, and $c_B$, there is a unique $p_A$ that solves (A1)’ or (A2)’, respectively. Denote the unique $p_A$ satisfying equations (A1) and (A2) by $p_A$ and $\bar{p}_A$, respectively.

Since the right-hand side of (A1)’ is positive and $f(.)<0$, we know that $p_A > 0$. Also, since $\frac{q-c_B}{z} > 1$ and $F\left( \frac{q-c_B}{z} \right) = 0$, we have $\bar{p}_A < q-c_B$. Since $f(\theta)$ is negative and $\frac{q}{q-z} > 1$, we have $0 < p_A < \bar{p}_A$.

We have the following three cases: $p_A \leq p_A ; \quad p_A < p_A < \bar{p}_A ; \quad \bar{p}_A \leq p_A$. 


Case 1: If $p_A \leq p_A^-$, (since $f(.) < 0$), we have $\frac{\partial \Pi_{B1}}{\partial p_B} \mid \frac{q - z}{z} p_A \geq 0$, and $\frac{\partial \Pi_{B2}}{\partial p_B} \mid \frac{q - z}{z} p_A > 0$.

Since $\frac{\partial \Pi_{B1}}{\partial p_B} \mid \frac{q - z}{z} p_A \geq 0$, $p_{B1}^*$, at which the peak point of $\Pi_{B1}$ occurs, is larger than or equal to $\frac{q - z}{z} p_A$. That is, $p_{B1}^*$ does not violate the constraint that $p_B \geq \frac{q - z}{z} p_A$ for $\Pi_{B1}$.

However, $p_{B2}^*$ violates the constraint that $p_B \leq \frac{q - z}{z} p_A$ for $\Pi_{B2}$. Because $\Pi_{B1}(\frac{q - z}{z} p_A) = \Pi_{B2}(\frac{q - z}{z} p_A)$, the overall profit function has only one peak, and firm B’s global optimum is given by the peak of $\Pi_{B1}$ and is achieved at $p_{B1}^*$.

Case 3: If $p_A \leq p_A$, then $\frac{\partial \Pi_{B1}}{\partial p_B} \mid \frac{q - z}{z} p_A < 0$, and $\frac{\partial \Pi_{B2}}{\partial p_B} \mid \frac{q - z}{z} p_A \leq 0$. A similar argument establishes that the global maximum is achieved at $p_B^* = p_{B2}^*$.

Case 2: If $p_A < p_A < p_A$, then $\frac{\partial \Pi_{B1}}{\partial p_B} \mid \frac{q - z}{z} p_A < 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} \mid \frac{q - z}{z} p_A > 0$. In this case, the global maximum is achieved at the kink, $p_B^* = \frac{q - z}{z} p_A$.

From the definition of $p_A$ and $p_A$, $\frac{\partial \Pi_{B1}}{\partial p_B} \mid \frac{q - z}{z} p_A = 0$, and $\frac{\partial \Pi_{B2}}{\partial p_B} \mid \frac{q - z}{z} p_A = 0$.

Thus, the strict inequalities of Case 2 can be extended to become weak inequalities, and firm B’s optimal price is continuous in $p_A$. Q. E. D

**Proof of Lemma 2:**

Firm A’s profit function is composed of two underlying profit functions, $\Pi_{A1}$ $(p_A; z) = F(\frac{p_A}{z}) (p_A - c_A)$ if $p_A \leq \bar{p}_A$ and $\Pi_{A2}(p_A) = F(\frac{p_A + p_A}{q}) (p_A - c_A)$ if $p_A \geq \bar{p}_A$. The two profit functions intersect at $p_A = \bar{p}_A$.

Under the monotonic hazard rate condition, each of these two profit functions is strictly quasi-concave in $p_A$. Let $p_{A1}^*$ and $p_{A2}^*$ denote the unconstrained optimal prices...
for the two profit functions, respectively. The first-order condition of these profit functions are as follows:

\[
\frac{\partial \Pi_{A1}}{\partial p_A} = F\left(\frac{p_A}{z}\right) + f\left(\frac{p_A}{z}\right) (p_A - c_A) \frac{1}{z} = 0 \tag{A3}
\]

\[
\frac{\partial \Pi_{A2}}{\partial p_A} = F\left(\frac{p_A + p_B}{q}\right) + f\left(\frac{p_A + p_B}{q}\right) \left(1 + \frac{dp_A}{dp_B}\right) (p_A - c_A) \frac{1}{q} = 0 \tag{A4}
\]

Let us evaluate these two first-order conditions at the kink \( \bar{p}_A \), taking into account the fact that firm B sets its price equal to \( \frac{q-z}{z} \bar{p}_A \).

\[
\frac{\partial \Pi_{A1}}{\partial p_A} \bigg|_{\bar{p}_A} = F\left(\frac{\bar{p}_A}{z}\right) + f\left(\frac{\bar{p}_A}{z}\right) (\bar{p}_A - c_A) \frac{1}{z} = 0 \tag{A3'}
\]

\[
\frac{\partial \Pi_{A2}}{\partial p_A} \bigg|_{\bar{p}_A} = F\left(\frac{\bar{p}_A}{z}\right) + f\left(\frac{\bar{p}_A}{z}\right) \left(1 + \frac{dp_A}{dp_B}\right) (\bar{p}_A - c_A) \frac{1}{q} = 0 \tag{A4'}
\]

(A3’) and (A4’) can be written in the following way.

\[
\bar{p}_A - c_A = -z \frac{F\left(\frac{\bar{p}_A}{z}\right)}{f\left(\frac{\bar{p}_A}{z}\right)} \tag{A3'’}
\]

\[
(1 + \frac{dp_B}{dp_A}) (\bar{p}_A - c_A) = -q \frac{F\left(\frac{\bar{p}_A}{z}\right)}{f\left(\frac{\bar{p}_A}{z}\right)} \tag{A4'’}
\]

Since \(-F(\theta)/f(\theta)\) is decreasing in \(\theta\), the right-hand side of (A3’’) and (A4’’) is monotonically increasing in \(z\). Thus, there is a unique \(z\) that solves (A3’’) or (A4’’), respectively. These unique \(z\)'s will be denoted by \(z_1\) and \(z_2\), respectively.

By comparing (A3’) and (A4’), we determine the relative size of \(z_1\) and \(z_2\). As Lemma 1 shows, for \(p_A \geq \bar{p}_A\), products A and B would be strict complements, implying that \(\frac{dp_A}{dp_B} < 0\). Since \(\frac{dp_A}{dp_B} < 0\) and \(q > z\), \(\frac{1}{z}\) is larger than \(1 + \frac{dp_B}{dp_A}\frac{1}{q}\). Also, \(f(\theta)\) is negative. Thus, the left-hand side of expression in (A4’) is larger than that of (A3’) for the same \(z\). Thus, \(z_1 < z_2\), and we have the following three cases: \(z \leq z_1; z_1 < z < z_2; z_2 \leq z\).
Product Boundary

Case 1: If $z \leq z_1$, then we have $\frac{\partial \Pi_{A1}}{\partial p_A} \mid p_A = \bar{p}_A \geq 0$, and $\frac{\partial \Pi_{A2}}{\partial p_A} \mid p_A > 0$. The peak point for $\Pi_{A2}$ occurs at $p_{A2}^*$, which is larger than $\bar{p}_A$. Since $\Pi_{A1}(\bar{p}_A; z) = \Pi_{A2}(\bar{p}_A)$ for all $z$, firm A’s global optimum is given by the peak of $\Pi_{A2}$ and is achieved at $p_{A2}^*$. Because $\frac{\partial \Pi_{A2}}{\partial p_A} \mid p_A > 0$, we have $p_{A2}^* > \bar{p}_A$.

Case 3: If $z_2 \leq z$, then $\frac{\partial \Pi_{A1}}{\partial p_A} \mid p_A < 0$, and $\frac{\partial \Pi_{A2}}{\partial p_A} \mid p_A \leq 0$. A similar argument establishes that $p_{A1}^*$ is firm A’s globally optimal price, and $p_{A1}^* < \bar{p}_A$.

Case 2: If $z_1 < z < z_2$, then $\frac{\partial \Pi_{A1}}{\partial p_A} \mid p_A < 0$, and $\frac{\partial \Pi_{A2}}{\partial p_A} \mid p_A > 0$. Thus, the overall profit function has two peaks, one at $p_{A1}^*$ and the other at $p_{A2}^*$, where $p_{A1}^* < \bar{p}_A < p_{A2}^*$, and we need to compare $\Pi_{A1}(p_{A1}^*; z)$ and $\Pi_{A2}(p_{A2}^*)$ for $z \in (z_1, z_2)$ to ascertain firm A’s globally optimal price. Notice that $\Pi_{A1}(p_{A1}^*; z)$ is continuously increasing in $z$, while $\Pi_{A2}(p_{A2}^*)$ is independent of $z$.

From the definition of $z_1$, $\Pi_{A1}(p_{A1}; z_1)$ is maximized at $\bar{p}_A$. Thus, $\Pi_{A1}(p_{A1}^*; z_1) = \Pi_{A1}(\bar{p}_A; z_1) = \Pi_{A2}(\bar{p}_A) < \Pi_{A2}(p_{A2}^*)$, where the second equality holds because $D_{A1}(\bar{p}_A) = D_{A2}(\bar{p}_A)$.

From the definition of $z_2$, $\Pi_{A2}(p_{A2})$ is maximized at $\bar{p}_A$. Thus, $\Pi_{A2}(p_{A2}^*) = \Pi_{A2}(\bar{p}_A) = \Pi_{A1}(\bar{p}_A; z_2) < \Pi_{A1}(p_{A1}^*; z_2)$, where the second equality holds because $D_{A1}(\bar{p}_A) = D_{A2}(\bar{p}_A)$.

Combining the above inequalities yields $\Pi_{A1}(p_{A1}^*; z_1) < \Pi_{A2}(p_{A2}^*) < \Pi_{A1}(p_{A1}^*; z_2)$. Since $\Pi_{A1}(p_{A1}^*; z)$ is continuously increasing in $z$, and $\Pi_{A2}(p_{A2}^*)$ is independent of $z$, there exists a unique $z^+$ strictly between $z_1$ and $z_2$ such that $\Pi_{A1}(p_{A1}^*; z^+) = \Pi_{A2}(p_{A2}^*)$; for $z < z^+$, $\Pi_{A1}(p_{A1}^*; z) < \Pi_{A2}(p_{A2}^*)$; for $z > z^+$, $\Pi_{A1}(p_{A1}^*; z) > \Pi_{A2}(p_{A2}^*)$. 

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These inequalities also hold if \( z \) takes on extreme values: if \( z = 0 \), then \( \Pi_{A1}(p_{A1}^*, z) = 0 < \Pi_{A2}(p_{A2}^*) \); if \( z \geq q - c_B \), then the demand for product (A+B) is zero, and \( \Pi_{A1}(p_{A1}^*, z) > \Pi_{A2}(p_{A2}^*) = 0 \). Thus, \( 0 < z^+ < q - c_B \).

Since \( z^+ \) lies between \( z_1 \) and \( z_2 \), by combining cases 1-3, we know that firm A’s global optimum is achieved at \( p_{A1}^* \) if \( z \geq z^+ \) and is achieved at \( p_{A2}^* \) if \( z \leq z^+ \). Finally the inequality that \( p_{A1}^*(z^+) < p_{A2}^* \) follows from \( p_{A1}^*(z) < \overline{p}_A < p_{A2}^* \) at \( z^+ \). Q. E. D

**Proof of Lemma 3**

By definition, \( p_{A1}^* \) satisfies (A1), and \( p_{A2}^* \) satisfies (A3). By comparing (A1) and (A3), we find that \( p_{A1}^* < p_{A2}^* \) if and only if \( z > \frac{c_A}{c_A + c_B} \). Q. E. D

**Proof of Proposition 4**

Since firm A is indifferent between \( p_{A2}^* \) and \( p_{A1}^*(z^+) \) at \( z^+ \), its profit function is continuous at \( z^+ \). After \( z^+ \), firm A maximizes \( F(p_A/z)(p_A - c_A) \), and its profit increases in \( z \) after \( z^+ \).

We will show that firm B’s profit jumps up at \( z^+ \). If \( z < z^+ \), firm B’s demand quantity is \( F(\frac{p_{A1}^* + p_{B2}^*}{q}) \), and we have \( \frac{p_{A2}^*}{z} > \frac{p_{B2}^*}{q - z} \). At \( z = z^+ \), firm A sets its price at \( p_{A1}^*(z^+) \). Suppose that firm B still sets its price at the same level, \( p_{B2}^* \). If \( \frac{p_{A1}^*(z^+)}{z} > \frac{p_{B2}^*}{q - z} \), or equivalently if \( \theta_A > \theta_B \), then firm B’s demand is \( F(\frac{p_{A1}^*(z^+)+p_{B2}^*}{q}) \), which is larger than \( F(\frac{p_{A2}^*+p_{B2}^*}{q}) \) because \( p_{A1}^*(z^+) < p_{A2}^* \). If \( \frac{p_{A1}^*(z^+)}{z} < \frac{p_{B2}^*}{q - z} \), or equivalently if \( \theta_A < \theta_B \), then firm B’s demand is \( F(\frac{p_{A2}^*+p_{B2}^*}{q}) \), which is larger than \( F(\frac{p_{A1}^*+p_{B2}^*}{q}) \) because \( \frac{p_{A1}^*}{z} > \frac{p_{B2}^*}{q - z} \). Thus, regardless of the relative size of \( \theta_A \) and \( \theta_B \), firm B’s demand jumps up at \( z^+ \), and as a result, its profit jumps up.

We show that firm B’s profit decreases as \( z \) increases between \( z^+ \) and \( \frac{p_{A1}^*}{c_A + c_B} \). If \( z \) is in this range, since \( \theta_A = \theta_B \), firm B’s profit function becomes \( \Pi_B(p_B) = F(\frac{p_{A1}^*}{q})(p_B - c_B) \),
which is decreasing in \( p_A \). Thus, since \( p_A \) increases in \( z \), it follows that firm B’s profit decreases in \( z \) between \( z^+ \) and \( \frac{c_A}{c_A + c_B} \).

After \( z^* \), firm B behaves as if it is a monopolist selling a product of quality \((q-z)\), and its profit decreases in \( z \).

Since firm B’s profit jumps up at \( z^+ \), over a certain range, firm B's profit at \( z > z^+ \) exceeds its profits at \( z \in [0, z^+] \). Q. E. D

**Proof of Proposition 5**

First, we show that the sum of the two firms' profits increases in \( z \) between \( z^+ \) and \( \frac{qc_A}{c_A + c_B} \). If \( z \) lies in this range, the vertically integrated firm maximizes \( qF(x_{A+B})( x_{A+B} - \frac{c_A + c_B}{q} ) \) by choosing \( x_{A+B} \). However, if firms A and B set their prices individually, we have \( \theta_A = \theta_B \), and firm A chooses \( \theta_A^* \) such that it maximizes \( zF(\theta_A)(\theta_A - \frac{c_A}{z}) \). If \( z < \frac{q c_A}{c_A + c_B} \), or equivalently if \( \frac{c_A + c_B}{q} < \frac{c_A}{z} \), (since a firm’s optimal cut-off point increases in its marginal cost), then \( x_{A+B}^* < \theta_A^* \). That is, \( \theta_A^* \) is too high in terms of the joint profits. As \( z \) increases, \( \theta_A^* \) becomes closer to \( x_{A+B}^* \), and their joint profits increase.

Second, we show that the sum of the two firm’s profits increases in \( z \) after \( \frac{qc_A}{c_A + c_B} \). If \( z > \frac{qc_A}{c_A + c_B} \), the two profit functions are independent of each other, and by the envelope theorem, we obtain

\[
\frac{\partial (\Pi_A + \Pi_B)}{\partial z} = F(\theta_A^*)(\theta_A^*) - F(\theta_B^*)(\theta_B^*).
\]

Firm A’s optimal \( \theta_A \) maximizes \( F(\theta_A)(\theta_A - \frac{c_A}{z}) \), and firm B’s optimal \( \theta_B \) maximizes \( F(\theta_B)(\theta_B - \frac{c_B}{z}) \). (Note that \( F(\theta_A^*)(\theta_A^*) \) is revenue for a firm maximizing \( F(\theta_A)(\theta_A - \frac{c_A}{z}) \)). As a firm’s marginal cost gets lower, its revenue gets higher. That is, if \( \frac{c_A}{z} < \frac{c_B}{q-z} \), or equivalently if \( z > \frac{qc_A}{c_A + c_B} \), then \( F(\theta_A^*)(\theta_A^*) > F(\theta_B^*)(\theta_B^*) \). Thus,

\[
\frac{\partial (\Pi_A + \Pi_B)}{\partial z} = F(\theta_A^*)(\theta_A^*) - F(\theta_B^*)(\theta_B^*) > 0. \quad \text{Q. E. D}
\]
Proof of Proposition 7

(a) If the two products are virtually strict complements, $z$ has no effect on the firms’ profits or on firm B’s incentive to increase $q$.

(b) If the two products are pseudo complements, firm A sets its price at $p^*_A$, and firm B sets its price such that $\theta_B = \theta_A$. Thus, firm B’s profit function becomes,

$$\Pi_B = F(\theta_A) (p_B - c_B)$$

$$= F(\theta_A) ((q-z)\theta_A - c_B)$$

$$\frac{\partial \Pi_B}{\partial q}$$ is $F(\theta_A)\theta_A$. Thus

$$\frac{\partial^2 \Pi_B}{\partial q^2} = [f(\theta_A)\theta_A + F(\theta_A)] \frac{\partial \theta_A}{\partial z}.$$ In the case of pseudo complements, $p_A$ satisfies the following first-order condition,

$$F\left(\frac{p_A}{z}\right) + f\left(\frac{p_A}{z}\right)(p_A - c_A) = 0.$$ This first-order condition implies that $F(\theta_A) + f(\theta_A)\theta_A = f(\theta_A)c_A \frac{1}{z} < 0$. From section 3.5, we know that $\frac{\partial \theta_A}{\partial z} < 0$. Consequently,

$$\frac{\partial^2 \Pi_B}{\partial z^2} = [f(\theta_A)\theta_A + F(\theta_A)] \frac{\partial \theta_A}{\partial z} > 0.$$ That is, an increase in $z$ strengthens firm B’s incentive to increase $q$.

(c) In the case of virtual independency, firm B’s demand becomes independent of $p_A$, and its profit function is $F(\theta_B)((q-z)\theta_B - c_B)$, implying that the important variable is the size of $k = (q-z)$, not the individual values of $q$ and $z$.

We show that $\Pi_B(\cdot)$ is convex in $k$, i.e., $\lambda \Pi_B(k_1) + (1-\lambda)\Pi_B(k_2) \geq \Pi_B(\lambda k_1 + (1-\lambda)k_2)$. Let $\theta_1$, $\theta_2$ and $\theta_3$ denote the optimal cutoff points for $k_1$, $k_2$ and $\lambda k_1 + (1-\lambda)k_2$, respectively.

$$\Pi_B(\lambda k_1 + (1-\lambda)k_2)$$

$$= F(\theta_3)((\lambda k_1 + (1-\lambda)k_2)\theta_3 - c_B)$$

$$= \lambda F(\theta_3)(k_1\theta_3 - c_B) + (1-\lambda)F(\theta_3)(k_2\theta_3 - c_B)$$

$$\leq \lambda F(\theta_1)(k_1\theta_1 - c_B) + (1-\lambda)F(\theta_2)(k_2\theta_2 - c_B)$$

$$= \lambda \Pi_B(k_1) + (1-\lambda)\Pi_B(k_2),$$

where the inequality follows from the definition that $\theta_1$ is the optimal cut-off point under $k_1$ and $\theta_2$ is the optimal cut-off point under $k_2$.

Since $\Pi_B(\cdot)$ is convex with respect to $(q-z)$, $\frac{\partial^2 \Pi_B}{\partial z^2}$ is negative. Thus, an increase in $z$ weakens firm B’s incentive to increase $q$. Q. E. D
Figure 1. $\theta_A < \theta_{A+B} < \theta_B$

Figure 2. $\theta_B < \theta_{A+B} < \theta_A$
Figure 3. Regions of virtually independent products and virtually strict complements.

. $\theta_A < \theta_{A+B} < \theta_B$: virtually independent products

. $\theta_B < \theta_{A+B} < \theta_A$: virtually strict complements

Figure 4. Firm B’s demand curve in the case of a uniform distribution. As $p_A$ increases, $D_{B2}$ shifts in.
Figure 5. Firm B’s best response to $p_A$ in the case of a uniform distribution.

Figure 6. Product relationship between A and B
Figure 7. Firm A’s optimal price in the case of a uniform distribution.

Figure 8. Firm A and B’s profit level in the case of a uniform distribution